DESCRIPTION OF PLATE.

No. 1.—Aurora spectrum with krypton comparison. The yellow-green aurora line, λ 5578, is seen on the right in approximate coincidence with the strong krypton line.

No. 2.—Spectrum of nitrogen, in the cathode rays and atomic rays separately, seen on a background of dark space, which is only relatively dark. Note the development of the various members of the blue group in each case. The atomic rays give an intensity distribution more like the aurora.

No. 3.—Spectrum of negative glow above (second positive group of bands marked) and of dark space below (aluminium lines marked).

No. 4.—Spectrum of first negative layer. Nitrogen line marked.

No. 5.—Spectrum of the capillary of a nitrogen spectrum tube at low pressure, placed in comparison with aurora. Note similar development of bands.

No. 6.—Cathode rays and positive rays in a mixture of five parts helium and one part nitrogen. Helium lines marked.

No. 7.—Direct photograph of the discharge tube as used in obtaining spectra 2, 3, and 6. The dark space traversed by atomic rays (straight) and cathode rays (deflected). Below is the negative glow, then the positive column. The tube is photographed through ultra-violet glass. Scale about one-quarter.

Fourier's Series and Analytic Functions.

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1. *Introduction.*

1.1. In the theorems which follow we are concerned with functions $f(x)$ real for real x and integrable in the sense of Lebesgue. We do not, however, remain in the field of the real variable, for we suppose, in §§ 4 *et seq.*, that $f(x)$, or a function associated with $f(x)$, is analytic, or, at any rate, harmonic, in a region of the complex plane associated with the particular real value of x considered.

The Fourier's series considered are those associated with the interval $(0, 2\pi)$. If a is a point of the interval, we write

$$\phi(u) = \tfrac{1}{2}\{f(a+u)+f(a-u)-2s\} \qquad (0 < a < 2\pi), \qquad (1.11)$$

$$\phi(u) = \tfrac{1}{2}\{f(u)+f(2\pi-u)-2s\} \qquad (a = 0,\ a = 2\pi), \qquad (1.12)$$

where s is a constant.

We write

$$z = x+iy, \qquad w = u+iv \tag{1.13}$$

for the complex variables of which x and u are the real parts.

By a region Δ, or $\Delta(\delta, \alpha)$, in the plane of w, we mean a triangle defined by inequalities

$$0 \leqq \mathrm{R}(w) \leqq \delta, \quad |\arg w| \leqq \alpha \quad (\delta > 0,\ 0 < \alpha \leqq \tfrac{1}{2}\pi). \tag{1.14}$$

When we say that $\phi(w)$ *is analytic (or harmonic) and bounded in* Δ, we mean that it is an analytic (or harmonic) function regular at all points inside Δ, and that its modulus at all such points is less than a constant $\mathrm{K} = \mathrm{K}(\delta, \alpha)$. It is immaterial whether we suppose $\phi(w)$ regular at points on the boundary of Δ, other than the origin; for we can always replace Δ by a slightly smaller region of the same type. The origin is, in general, an essential singularity of $\phi(w)$.

In § 2 we prove two lemmas concerning derivatives of functions of a real variable. In § 3, remaining in the real domain, we prove a theorem concerning Fourier's series, of which our main result proves later to be a corollary. In § 4 we prove two lemmas concerning harmonic functions. In § 5 we consider our principal problem. *Given that* $\phi(w)$ *is harmonic and bounded in a region* Δ, *what is the necessary and sufficient condition that the Fourier's series of* $f(x)$ *should converge when* $x = a$? The answer to this question is contained in theorem 2. We also prove an analogous theorem in which, instead of supposing $\phi(w)$ bounded, we suppose that the integral

$$\iint |\phi'(w)|^2\, du\, dv, \tag{1.15}$$

taken over Δ, is convergent. In § 6 we add some notes relating to certain theorems of Lindelöf and Montel.

2. *Lemmas concerning derivatives of functions of a real variable.*

2.1. *Lemma* 1. *If* $\psi(u)$ *is a function of the real variable* u, *continuous and indefinitely differentiable for* $0 < u < \delta$, *and*

$$\psi(u) = o(1), \qquad \psi^{(p)}(u) = O\left(\frac{1}{u^p}\right), \tag{2.12}$$

when $u \to 0$, *then*

$$\psi^{p}(u) = o\left(\frac{1}{u^p}\right). \tag{2.13}$$

This is a special case of a known theorem.[*]

2.2. Our second lemma is of the same character, but is not actually included in any of the theorems proved in the memoir just referred to.

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[*] G. H. Hardy and J. E. Littlewood, "Contributions to the Arithmetic Theory of Series," 'Proc. London Math. Soc.' (2), vol. 11, pp. 411–478, 427, Theorem 8 (1913).

Lemma 2. *Suppose that* $\psi(u)$ *is a function of the real variable* u *which possesses a third derivative* $\psi'''(u)$ *for* $0 < u < \delta$, *and that*

$$\psi(u) = o(u^2), \qquad \psi'''(u) = O\left(\frac{1}{u}\right) \tag{2.21}$$

when $u \rightarrow 0$. *Then*

$$\psi''(u) \rightarrow 0 \tag{2.22}$$

when $u \rightarrow 0$.

Let $\xi = \zeta u$, where $0 < \zeta < \frac{1}{2}$, and suppose that $0 < (1+\zeta)u < \delta$. Then, by Taylor's Theorem,

$$\psi(u+\xi)+\psi(u-\xi)-2\psi(u) = \xi^2\psi''(u)+\tfrac{1}{6}\xi^3\{\psi'''(u+\theta_1\xi)-\psi'''(u-\theta_2\xi)\},$$

where $0 < \theta_1 < 1$, $0 < \theta_2 < 1$; and therefore

$$\begin{aligned}|\psi''(u)| &\leqq \frac{1}{\xi^2}(|\phi(u+\xi)| + |\phi(u-\xi)| + 2|\phi(u)|) \\ &\qquad + \frac{\xi}{6}(|\psi'''(u+\theta_1\xi)| + |\psi'''(u-\theta_2\xi)|) \\ &= \lambda + \mu, \end{aligned} \tag{2.23}$$

say. Since $\psi'''(u) = O\left(\frac{1}{u}\right)$, we have

$$|\mu| < \mathrm{H}\xi\left(\frac{1}{u+\theta_1\xi}+\frac{1}{u-\theta_2\xi}\right) < \frac{2\mathrm{H}\zeta}{1-\zeta},$$

where H is a constant; and we can therefore choose ζ so that

$$|\mu| < \tfrac{1}{2}\epsilon. \tag{2.24}$$

When ζ is fixed, we have

$$\lambda = \frac{1}{\zeta^2u^2}\{(1+\zeta)^2 o(u^2)+(1-\zeta)^2 o(u^2)+2o(u^2)\} = o(1)$$

so that

$$|\lambda| < \tfrac{1}{2}\epsilon, \tag{2.25}$$

if u is sufficiently small. From (2.23), (2.24), and (2.25) the conclusion follows.

The form of the proof is modelled on that used by Landau in the discussion of theorems of this character.*

3. *Preliminary theorem.*

3.1. THEOREM 1. *Suppose that* $\phi(u)$ *possesses a second derivative* $\phi''(u)$ *for* $0 < u < \delta$, *and that*

$$\phi''(u) = O\left(\frac{1}{u^2}\right). \tag{3.11}$$

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* See E. Landau, "Einige Ungleichungen für zweimal differentiierbare Funktionen," 'Proc. London Math. Soc.' (2), vol. 13, pp. 43–49 (1913); and 'Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie,' p. 54 (1916).

Then the necessary and sufficient condition that the Fourier's series of $f(x)$, for $x = a$, should converge to the sum s, is that

$$\phi(u) \to 0$$

when $u \to 0$ through positive values.

We make certain preliminary simplifications. We suppose, first, that $0 < a < 2\pi$; the modifications required when a is an end-point of the interval are trivial. We may then suppose $\delta < \operatorname{Min}(a, 2\pi - a)$.

We may suppose, without loss of generality, that $f = 0$, except in the interval $(a - \delta, a + \delta)$, for the behaviour of f outside this interval is irrelevant to the question of convergence. We may also suppose that $s = 0$. Thus

$$\phi(u) = \tfrac{1}{2}\{f(a+u)+f(a-u)\} \ (0 \leqq u \leqq \delta), \quad \phi(u) = 0 \ (u > \delta). \tag{3.13}$$

If the Fourier's series of $f(x)$ is

$$f(x) \sim A_0 + \sum_1^\infty A_n = \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx), \tag{3.14}$$

then that of $\phi(u)$ is

$$\phi(u) \sim A_0 + \sum_1^\infty (a_n \cos na + b_n \sin na) \cos nu, \tag{3.15}$$

and the Fourier's series of $f(x)$, for $x = a$, is that of $\phi(u)$ for $u = 0$. Our problem is therefore equivalent to the corresponding problem for this latter series.

We observe that (3.11) involves

$$\phi'(u) = O\left(\frac{1}{u}\right). \tag{3.16}$$

3.21. *The condition is sufficient.* This may be proved very simply in two different ways which are each of interest.

(a) It is sufficient to prove that

$$A_n = O\left(\frac{1}{n}\right). \tag{3.21}$$

For the series is summable (C1), by the classical theorems of Fejér and Lebesgue, and a series which is summable (C1), and satisfies (3.21), is necessarily convergent.*

It is enough then, to prove that

$$\int_0^\delta e^{niu}\phi(u)\,du = O\left(\frac{1}{n}\right). \tag{3.22}$$

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* L. Fejér, "Untersuchungen über Fourier'sche Reihen," 'Math. Ann.,' vol. 58, pp. 51–69 (1904); H. Lebesgue, "Recherches sur la convergence des séries de Fourier," 'Math. Ann.,' vol. 61, pp. 251–280 (1905); G. H. Hardy, "Theorems relating to the convergence and summability of slowly oscillating series," 'Proc. London Math. Soc.,' ser. 2, vol. 8, pp. 301–320 (1909). See also Ch. J. de la Vallée Poussin, 'Cours d'analyse infinitésimale,' vol. 2, ed. 2, pp. 154–162 (1912); H. S. Carslaw, 'Introduction to the theory of Fourier's series and integrals,' vol. 1, ed. 2, pp. 234–243 (1921).

Since $\phi(u)$ is bounded, the integral is equal to

$$O\left(\frac{1}{n}\right)+\int_{1/n}^{\delta} e^{niu}\phi(u)\,du = O\left(\frac{1}{n}\right)+\left[\frac{e^{niu}\phi(u)}{ni}\right]_{1/n}^{\delta} + \left[\frac{e^{niu}\phi'(u)}{n^2}\right]_{1/n}^{\delta} - \frac{1}{n^2}\int_{1/n}^{\delta} e^{niu}\phi''(u)\,du$$

$$= O\left(\frac{1}{n}\right)+O\left(\frac{1}{n}\right)+O\left(\frac{1}{n}\right)+O\left(\frac{1}{n^2}\int_{1/n}^{\delta}\frac{du}{u^2}\right) = O\left(\frac{1}{n}\right),$$

by (3.11) and (3.16).

(*b*) A sufficient condition for convergence is that

$$\mathrm{I}(\epsilon) = \int_{\epsilon}^{\delta} \frac{|\phi(u+\epsilon)-\phi(u)|}{u}\,du \to 0, \tag{3.23}$$

when $\epsilon \to 0$ by positive values.* But

$$\phi(u+\epsilon)-\phi(u) = \epsilon\phi'(u+\theta\epsilon) \quad (0 < \theta < 1)$$

and

$$\phi'(u) = o\left(\frac{1}{u}\right),$$

by Lemma 1. Hence

$$\mathrm{I}(\epsilon) = o\left(\epsilon\int_{\epsilon}^{\delta}\frac{du}{u(u+\theta\epsilon)}\right) = o\left(\epsilon\int_{\epsilon}^{\delta}\frac{du}{u^2}\right) = o(1),$$

which proves (3.23).

3.3. *The condition is necessary.* We denote by $\Phi(u)$ Riemann's function

$$\Phi(u) = \tfrac{1}{2}\mathrm{A}_0u^2+\sum_{1}^{\infty}\frac{\mathrm{A}_n}{n^2}(1-\cos nu) \quad (0 < u < \delta), \tag{3.31}$$

and by $\phi_2(u)$ the function

$$\phi_2(u) = \int_0^u dy\int_0^y \phi(t)\,dt \quad (0 < u < \delta), \tag{3.32}$$

These two functions are identical. For the Fourier's series of $\phi(u)$ is convergent when $0<u<\delta$, since $\phi(u)$ has a second derivative for such values of u, and therefore† $\Phi(u)$ has, for such values of u, a generalised second derivative equal to $\phi(u)$. But $\phi_2(u)$ has an ordinary second derivative, and *a fortiori* a generalised second derivative, also equal to $\phi(u)$. Hence

$$\Psi(u) = \Phi(u)-\phi_2(u)$$

has, for such values of u, a generalised second derivative which is everwhere zero; and is therefore‡ a linear function $\mathrm{C}+\mathrm{D}u$. If now the series is convergent for $u = 0$, we have* $\Phi(u) = o(u^2) = o(u)$, and it is evident from (3.32) that $\phi_2(u) = o(u)$. Hence $C + Du = o(u)$, which is only possible if $C = D = 0$. Thus $\phi_2(u) = \Phi(u)$, and so

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* H. Lebesgue, *loc. cit.*, p. 263, and 'Leçons sur les séries trigonométriques,' pp. 59, 64; Ch. J. de la Vallée Poussin, *loc. cit.*, p. 150.

† By 'Riemann's First Theorem,' De la Vallée Poussin, *loc. cit.*, p. 172.

‡ By 'Schwarz's Lemma,' De la Vallée Poussin, *loc. cit.*, p. 171.

$$\phi_2(u) = o(u^2). \tag{3.33}$$

Also
$$\phi_2'''(u) = \phi'(u) = O\left(\frac{1}{u}\right), \tag{3.34}$$

by (3.16). From (3.33) and (3.34) it follows, by Lemma 2, that

$$\phi_2''(u) = \phi(u) = o(1),$$

which completes the proof of the theorem.

4. *Lemmas concerning harmonic functions.*

4.1. *Lemma* 3.—*Suppose that* $u(x, y) = u(r, \theta)$ *is a real harmonic function regular for* $r < R$, *and that*

$$|u| < M \qquad (r \leqq R). \tag{4.11}$$

Then

$$\left|\left(\frac{\partial^{p+q}u}{\partial^p x\, \partial^q y}\right)_0\right| \leqq \frac{2(p+q)!\, M}{R^{p+q}}, \tag{4.12}$$

the suffix indicating a value taken at the origin.

We have, by the familiar formula of Poisson,

$$u(x, y) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi-\theta) + r^2}\, u(R, \phi)\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left\{\frac{R}{R - e^{i\phi}(x - iy)} + \frac{R}{R - e^{-i\phi}(x + iy)} - 1\right\} u(R, \phi)\, d\phi.$$

Hence

$$\left(\frac{\partial^{p+q}u}{\partial^p x \partial^q y}\right)_0 = \frac{(p+q)!}{2\pi R^{p+q}}\int_0^{2\pi} \{(-i)^q e^{(p+q)i\phi} + i^q e^{-(p+q)i\phi}\}\, u(R, \phi)\, d\phi,$$

and the result follows immediately.

4.2. *Lemma* 4.—*Suppose that* $\phi(w)$ *is harmonic and bounded in a region* Δ. *Then*

$$\phi^{(p)}(u) = \left(\frac{d}{du}\right)^p \phi(u) = O\left(\frac{1}{u^p}\right) \tag{4.21}$$

when $u \to 0$ *by positive values.*

Let ξ be a small real value of w, and $R = \xi \sin\alpha$ the perpendicular from $w = \xi$ on to one of the sides of Δ through $w = 0$. The circle whose centre is $w = \xi$, and whose radius is R, lies entirely inside Δ if ξ is sufficiently small. Applying lemma 3, with $q = 0$, we obtain

$$\phi^{(p)}(\xi) = O\left(\frac{1}{R^p}\right) = O\left(\frac{1}{\xi^p}\right),$$

which proves the lemma.

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* Again by 'Riemann's First Theorem.'

5. *Proof of the main theorems.*

5.1. THEOREM 2.—*Suppose that $\phi(w)$ is harmonic and bounded in a region Δ. Then the necessary and sufficient condition that the Fourier series of $f(x)$, for $x = a$, should be convergent, is that $\phi(u) \to 0$ when $u \to 0$ through positive values.*

By Lemma 4, $\phi''(u) = O\left(\frac{1}{u^2}\right)$. The theorem is thus an immediate corollary of Theorem 1.

As a special case we have,

THEOREM 3.—*If $\phi(u)$ is the value, when $u = 0$, of the real component of a function $F(w)$ of the complex variable w, and $F(w)$ is analytic and bounded in a region Δ, then the necessary and sufficient condition for convergence is that $\phi(u) \to 0$.*

This is the form in which we originally proved the theorem. The idea of extending it to any bounded harmonic function was suggested to us by Dr. Marcel Riesz.

5.2. We next prove a theorem which resembles Theorem 3 in all respects save one. We omit the condition that $F(w)$ is *bounded* in Δ, substituting for this the condition that the integral (1.15) should be convergent. The condition has a simple geometrical meaning, namely that Δ is mapped, by the transformation $W = F(w)$, on a region of finite area in the plane of W.

THEOREM 4. *Suppose that $\phi(u) = \phi(u, 0)$ is the value, for $v = 0$, of the real part $\phi(u, v)$ of a function $F(w)$ analytic in Δ, and that the integral (1.15) is convergent. Then the necessary and sufficient condition that the Fourier's series of $f(x)$, for $x = a$, should converge to the sum s, is that $\phi(u) \to 0$ when $u \to 0$ through positive values.**

We have our proof of Theorem 4 on the following additional lemma.

5.2. *Lemma 5.—If $F(w)$ is analytic in Δ, and the integral (1.15) is convergent, then*

$$F^{(n)}(w) = o\left(\frac{1}{w^n}\right) \tag{5.21}$$

when $w \to 0$ in any manner inside a region Δ' interior to Δ. In particular

$$\phi^{(n)}(u) = o\left(\frac{1}{u^n}\right) \tag{5.22}$$

when $u \to 0$ through positive values.

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* It may be well to point out explicitly that the convergence of (1.15) does not in itself imply the boundedness of $F(w)$ in Δ. A simple example to the contrary is obtained by taking $F(w) = \log\log w$. An example in which $F(w)$ is bounded, but (1.15) is not convergent, is given by $F(w) = w^{ci}$, where c is positive. The integral (1.15) is equal to either of the integrals

$$\iint (\phi_u{}^2 + \phi_v{}^2)\, du\, dv, \quad \iint (\psi_u + \psi_v{}^2)\, du\, dv,$$

where $\phi + i\psi = F$. There is therefore no theorem more general than 4 and related to 4 as 2 is to 3.

We require (5.21) only when w is real, and state our proof for this case. The modifications necessary in the general case are trivial and may be left to the reader.

Suppose then that u is real, and that C is the circle whose centre is $w = u$ and which touches the oblique sides of Δ. The radius of C is $\rho = u \sin \alpha$, and we have

$$\mathrm{F}(w) = \sum_{0}^{\infty} \frac{\mathrm{F}^{(n)}(u)}{n!}(w-u)^n = \sum_{0}^{\infty} \alpha_n (w-u)^n \tag{5.23}$$

for $|w-u| < \rho$. Also

$$\sum_{n=1}^{\infty} n|\alpha_n|^2 \rho^{2n} = \frac{1}{\pi}\iint |\mathrm{F}'(w)|^2 \, du \, dv, \tag{5.24}$$

the integration now being over the interior of C.* This integral tends to zero when $u \to 0$, since (1.15) is convergent. Hence

$$n|\alpha_n|^2 \rho^{2n} = \frac{n}{(n!)^2}|\mathrm{F}^{(n)}(u)|^2 (u \sin \alpha)^{2n} = o(1), \tag{5.25}$$

or

$$\mathrm{F}^{(n)}(u) = o\left(\frac{1}{u^n}\right);$$

which proves (5.21), for real w, and *a fortiori* (5.22).

5.23. The proof of Theorem 4 is now immediate. We have only to take $n = 2$ in (5.22), and the theorem becomes a corollary of Theorem 1.

Note on certain theorems of Lindelöf and Montel.

6.1. We were originally led to Theorem 2 by an argument which differs materially, as regards the proof of the *necessity* of (3.12), from that of § 3. Our original argument depended upon a very interesting theorem first proved by Montel.† This theorem, in its simplest form, may be stated as follows :—

Suppose that (i) $\phi(w)$ *is analytic and bounded in a region* Δ, *and* (ii) *that* $\phi(w) \to 0$ *when* $w \to 0$ *by positive values. Then* $\phi(w) \to 0$ *uniformly, when* $w \to 0$ *in any manner within any region* Δ' *interior to* Δ.

This theorem has been generalised by Montel himself and by Lindelöf,‡ in

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[*] See L. Fejér, "Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene," 'H. A. Schwarz Festschrift,' pp. 42–53 (47), (1914).

[†] P. Montel, "Sur les familles de fonctions analytiques qui admettent des valeurs exceptionnelles dans un domaine," 'Annales Scientifiques de l'École Normale Supérieure,' (3), vol. 29, pp. 487–535 (519), (1912).

[‡] E. Lindelöf, "Sur un principe général de l'analyse," 'Acta Soc. Fennicæ,' vol. 46, no. 4, pp. 1–35 (7), (1915). A different proof of what is substantially the theorem proved here by Lindelöf was given later by Hardy and Littlewood ("Abel's Theorem and its converse," 'Proc. London Math. Soc.,' ser. 2, vol. 18, pp. 205–235 (p. 218, lemma ϵ). (1917).

two directions. In the first place the hypothesis (i) may be replaced by the more general hypothesis that $\phi(w)$ does not assume either of two exceptional values. Secondly, we may consider, instead of the straight line of hypothesis (ii), any simple continuous path which leads to the origin from inside Δ.

We are concerned here only with the theorem in its simplest form. It seems worth while to point out how naturally it follows from some of the lemmas on which our preceding analysis depends.

Suppose that $|\phi(w)| < \mathrm{M}$ in Δ and that u is real. Then, if C is the circle of § 4.2, we have

$$|\phi^{(n)}(u)| = \left| \frac{n!}{2\pi i} \int_{\mathrm{C}} \frac{\phi(w)}{(w-u)^{n+1}} \right| \leqq \frac{n!\,\mathrm{M}}{(u \sin\alpha)^n}. \tag{6.11}$$

Thus
$$\phi(u) = o(1), \qquad \phi^{(n)}(u) = O\left(\frac{1}{u^n}\right),$$

for every n, and therefore, by Lemma 1,

$$\phi^{(n)}(u) = o\left(\frac{1}{u^n}\right). \tag{6.12}$$

Now let $\alpha' < \alpha$ be the angle of Δ' corresponding to α, and C′ the circle of radius $u \sin\alpha'$ corresponding to C. If w lies within C′ we have

$$\phi(w) = \sum_0^\infty \frac{\phi^{(n)}(u)}{n!}(w-u)^n = \sum_0^p + \sum_{p+1}^\infty = \phi_1 + \phi_2, \tag{6.13}$$

say. From (6.11) we deduce

$$|\phi_2| \leqq \sum_{p+1}^\infty \frac{1}{n!} \cdot \frac{n!\,\mathrm{M}}{(u\sin\alpha)^n} \cdot (u \sin\alpha')^n = \mathrm{M}\left(\frac{\sin\alpha'}{\sin\alpha}\right)^{p+1} \frac{\sin\alpha}{\sin\alpha - \sin\alpha'},$$

which tends to zero when $p \to \infty$. We have therefore

$$|\phi_2| < \tfrac{1}{2}\epsilon \tag{6.14}$$

if only p is sufficiently large. When p is fixed, every term of ϕ_1 is of the form

$$o\left(\frac{1}{u^n}\right) O(u^n) = o(1)$$

in virtue of (6.12), so that

$$|\phi_1| < \tfrac{1}{2}\epsilon \tag{6.15}$$

if u is sufficiently small. From (6.13), (6.14), and (6.15) the conclusion follows.

6.2. The direct analogue of Montel's theorem for *harmonic* functions is false, as appears at once from the trivial example $\phi = \arctan(v/u)$. A modification of the argument of 6.1 leads, however, to the following result:—

Suppose that $\phi(u, v)$ is a real harmonic function, regular and bounded in Δ, *and that* $\phi \to 0$ *when* $w \to 0$ *along either of two radii vectores interior to* Δ. *Then* $\phi \to 0$ *uniformly when* $w \to 0$ *in any manner within any region* Δ' *interior to* Δ.

6.3. Similar arguments, based upon Lemma 5, lead to the conclusion that *in Montel's theorem of* 6.1, *the hypothesis that* $\phi(w)$ *is bounded may be replaced by the hypothesis that the integral* (1.15) *is convergent.* A direct proof of this theorem is not without interest, but the actual result is included in the more general theorems referred to in 6.1.*

On the Free Transverse Vibrations of a Uniform Circular Disc Clamped at its Centre; and on the Effects of Rotation.

By R. V. SOUTHWELL, M.A.

(Communicated by Prof. H. Lamb, F.R.S. Received September 23, 1921.)

INTRODUCTION AND SUMMARY.

§ 1. A thin circular disc, entirely unconstrained, can vibrate transversely in modes which may be characterized by any number of nodal concentric circles, crossed by uniformly spaced nodal diameters. The natural frequencies, for a disc of uniform thickness, have been calculated by Kirchhoff,† and a recent investigation by Prof. Lamb and the present author‡ has shown the amount by which they are increased when the disc rotates at any given speed about its axis.

This extension of Kirchhoff's analysis was suggested by published accounts of failures in turbine discs, the circumstances of which indicated that they had their origin in forced vibrations, intensified by resonance.§ It was realised that the results would be only indirectly applicable to design, mainly because in practice discs are constructed with curved profiles (the thickness increasing from the rim to the boss), but also because the calculated

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* For $\phi(w)$ covers a finite area when w covers Δ, and there are therefore infinitely many values which $\phi(w)$ does not take.

† "Ueber die Schwingungen einer elastischen Scheibe," 'Crelle's Journal,' vol. 40 (1850), and 'Pogg. Ann.,' vol. 81 (1850). The essentials of the analysis are reproduced in Lord Rayleigh's 'Theory of Sound,' vol. 1, §§ 218, 219.

‡ 'Roy. Soc. Proc.,' A, vol. 99, pp. 272–280 (1921). Throughout the present investigation this paper will be quoted by the abbreviated title "Spinning Discs," and its equations by their number with an A affixed: thus equation (34A) below is equation (34) of the "Spinning Discs" paper.

§ *Cf.* K. Baumann, "Some Recent Developments in Large Steam Turbine Practice," Journ. Inst. Elect. Eng.,' vol. 59, pp. 619, 620 (1921).